Basics of Numerical Optimization: Iterative Methods

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Logistics

- Project grouping
  * Proposal due: Oct 23
  * Proposal template: https://nips.cc/Conferences/2020/PaperInformation/StyleFiles

- Colab purchase

Find global minimum

$$\min_{\boldsymbol{x}} \; f(\boldsymbol{x})$$

Grid search: incurs $O\left(\varepsilon^{-n}\right)$ cost

Smart search

> **1st-order necessary condition**: Assume f is 1st-order differentiable at $\boldsymbol{x}_0$. If $\boldsymbol{x}_0$ is a local minimizer, then $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$.

$\boldsymbol{x}$ with $\nabla f(\boldsymbol{x}) = \mathbf{0}$: **1st-order stationary point** (1OSP)

> **2nd-order necessary condition**: Assume $f(\boldsymbol{x})$ is 2-order differentiable at $\boldsymbol{x}_0$. If $\boldsymbol{x}_0$ is a local min, $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}_0) \succeq \mathbf{0}$.

$\boldsymbol{x}$ with $\nabla f(\boldsymbol{x}) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}) \succeq \mathbf{0}$: **2nd-order stationary point** (2OSP)

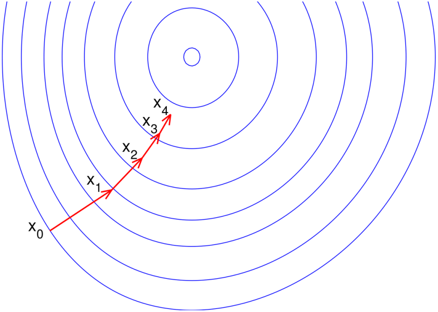
Smart search

$\boldsymbol{x}$ with $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$: **1st-order stationary point** (1OSP)
$\boldsymbol{x}$ with $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ and $\nabla^2 f(\boldsymbol{x}) \succeq \boldsymbol{0}$: **2nd-order stationary point** (2OSP)

- **Analytic method**: find 1OSP's using gradient first, then study them using Hessian — for simple functions! e.g., $f(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$, or $f(x, y) = x^2 y^2 - x^3 y + y^2 - 1$)
- **Iterative methods**: find 1OSP's/2OSP's by making consecutive small movements

This lecture: **iterative methods**

Iterative methods



Credit: aria42.com

Illustration of iterative methods on the contour/levelset plot (i.e., the function assumes the same value on each curve)

Two questions: what direction to move, and how far to move

Two possibilities:

- **Line-search methods**: direction first, size second
- **Trust-region methods**: size first, direction second

Outline

Framework of line-search methods

A generic line search algorithm

Input: initialization $\boldsymbol{x}_0$, stopping criterion (SC), $k = 1$

1: **while** SC not satisfied **do**

2: choose a direction $\boldsymbol{d}_k$

3: decide a step size t_k

4: make a step: $\boldsymbol{x}_k = \boldsymbol{x}_{k-1} + t_k \boldsymbol{d}_k$

5: update counter: $k = k + 1$

6: **end while**

Four questions:

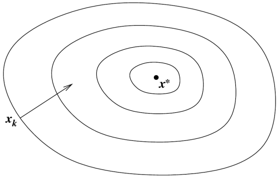
- How to choose direction $\boldsymbol{d}_k$?
- How to choose step size t_k?
- Where to initialize?
- When to stop?

We want to decrease the function value toward global minimum...
shortsighted answer: find a direction to decrease most rapidly

for any fixed $t > 0$, using 1st order Taylor expansion

$$f(\boldsymbol{x}_k + t\boldsymbol{d}_{k+1}) - f(\boldsymbol{x}_k) \approx t \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}_{k+1} \rangle$$

$$\min_{\|\boldsymbol{v}\|_2 = 1} \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{v} \rangle \implies \boldsymbol{v} = -\frac{\nabla f(\boldsymbol{x}_k)}{\|\nabla f(\boldsymbol{x}_k)\|_2}$$



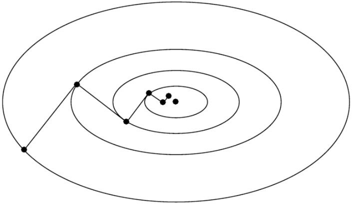
Set $\boldsymbol{d}_k = -\nabla f(\boldsymbol{x}_k)$

gradient/steepest descent: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t\nabla f(\boldsymbol{x}_k)$

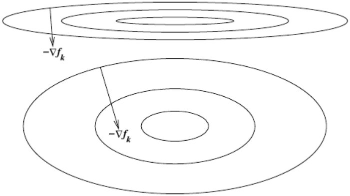
Gradient descent

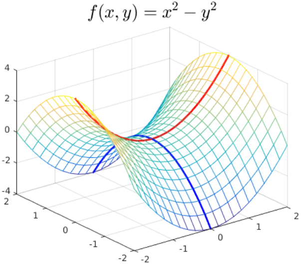
$$\min_{\boldsymbol{x}} \; \boldsymbol{x}^\intercal \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x}$$

typical zig-zag path



conditioning affects the path length





- remember direction curvature?
  $\boldsymbol{v}^\intercal \nabla^2 f(\boldsymbol{x}) \, \boldsymbol{v} = \frac{d^2}{dt^2} f(\boldsymbol{x} + t\boldsymbol{v}) \Big|_{t=0}$
- large curvature $\leftrightarrow$ narrow valley
- directional curvatures encoded in the Hessian

How to choose a search direction?

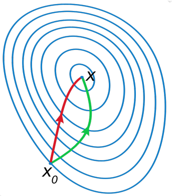
We want to decrease the function value toward global minimum...

shortsighted answer: find a direction to decrease most rapidly

farsighted answer: find a direction based on both gradient and Hessian

for any fixed $t > 0$, using 2nd-order Taylor expansion

$$f(\boldsymbol{x}_k + t\boldsymbol{v}) - f(\boldsymbol{v}) \approx t \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{v} \rangle + \frac{1}{2} t^2 \langle \boldsymbol{v}, \nabla^2 f(\boldsymbol{x}_k) \boldsymbol{v} \rangle$$



grad desc: green; Newton: red

minimizing the right side

$$\Longrightarrow \boldsymbol{v} = -t^{-1} \left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$$

Set $\boldsymbol{d}_k = \left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$

Newton's method: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t \left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$,

t can set to be 1.

Why called Newton's method?

Newton's method: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t\left[\nabla^2 f\left(\boldsymbol{x}_k\right)\right]^{-1} \nabla f\left(\boldsymbol{x}_k\right)$,

Recall Newton's method for root-finding

$$x_{k+1} = x_k - f'\left(x_n\right) f\left(x_n\right)$$

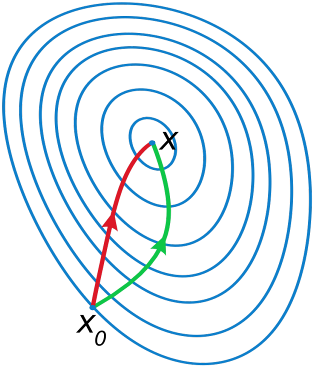
Newton's method for solving nonlinear system $f\left(\boldsymbol{x}\right) = \mathbf{0}$

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \left[\boldsymbol{J}_f\left(\boldsymbol{x}_n\right)\right]^{\dagger} \boldsymbol{f}\left(\boldsymbol{x}_n\right)$$

Newton's method for solving $\nabla f\left(\boldsymbol{x}\right) = \mathbf{0}$

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \left[\nabla^2 f\left(\boldsymbol{x}_n\right)\right]^{-1} \nabla \boldsymbol{f}\left(\boldsymbol{x}_n\right)$$

How to choose a search direction?



grad desc: green; Newton: red

Newton's method take fewer steps

nearsighted choice: cost $O(n)$ per step

gradient/steepest descent:
$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t\nabla f(\boldsymbol{x}_k)$

farsighted choice: cost $O(n^3)$ per step

Newton's method: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t\left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$,

Implication: The plain Newton never used for large-scale problems. More on this later ...

Problems with Newton's method

Newton's method: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t\left[\nabla^2 f\left(\boldsymbol{x}_k\right)\right]^{-1} \nabla f\left(\boldsymbol{x}_k\right)$,

for any fixed $t > 0$, using 2nd-order Taylor expansion

$$f\left(\boldsymbol{x}_k + t\boldsymbol{v}\right) - f\left(\boldsymbol{v}\right) \approx t\left\langle \nabla f\left(\boldsymbol{x}_k\right), \boldsymbol{v}\right\rangle + \frac{1}{2}t^2\left\langle \boldsymbol{v}, \nabla^2 f\left(\boldsymbol{x}_k\right)\boldsymbol{v}\right\rangle$$

minimizing the right side $\Longrightarrow \boldsymbol{v} = -t^{-1}\left[\nabla^2 f\left(\boldsymbol{x}_k\right)\right]^{-1} \nabla f\left(\boldsymbol{x}_k\right)$

- $\nabla^2 f\left(\boldsymbol{x}_k\right)$ may be non-invertible
- the minimum value is $-\frac{1}{2}\left\langle \nabla f\left(\boldsymbol{x}_k\right), \left[\nabla^2 f\left(\boldsymbol{x}_k\right)\right]^{-1} \nabla f\left(\boldsymbol{x}_k\right)\right\rangle$. If $\nabla^2 f\left(\boldsymbol{x}_k\right)$ not positive definite, may be positive

solution: e.g., modify the Hessian $\nabla^2 f\left(\boldsymbol{x}_k\right) + \tau \boldsymbol{I}$ with τ sufficiently large

How to choose step size?

$$\boldsymbol{x}_k = \boldsymbol{x}_{k-1} + t_k \boldsymbol{d}_k$$

- Naive choice: sufficiently small constant t for all k
- Robust and practical choice: back-tracking line search

Intuition for back-tracking line search:

- By Taylor's theorem,
  $f(\boldsymbol{x}_k + t\boldsymbol{d}_k) = f(\boldsymbol{x}_k) + t \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}_k \rangle + o(t \|\boldsymbol{d}_k\|_2)$ when t sufficiently small — $t \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}_k \rangle$ dictates the value decrease

- But we also want t large as possible to make rapid progress

- **idea**: find a large possible t^* to make sure
  $f(\boldsymbol{x}_k + t^*\boldsymbol{d}_k) - f(\boldsymbol{x}_k) \le ct^* \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}_k \rangle$ (**key condition**) for a chosen parameter $c \in (0, 1)$, and no less

- **details**: start from $t = 1$. If the **key condition** not satisfied, $t = \rho t$ for a chosen parameter $\rho \in (0, 1)$.

Back-tracking line search

A widely implemented strategy in numerical optimization packages

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Back-tracking line search

Input: initial $t > 0$, $\rho \in (0, 1)$, $c \in (0, 1)$

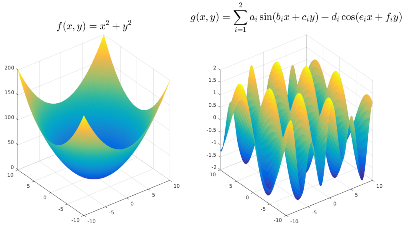
1: **while** $f(\boldsymbol{x}_k + t\boldsymbol{d}_k) - f(\boldsymbol{x}_k) \geq ct \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}_k \rangle$ **do**

2: $\quad t = \rho t$

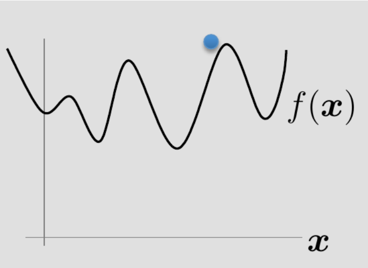
3: **end while**

Output: $t_k = t$.

Where to initialize?



convex vs. nonconvex functions



- **Convex**: most iterative methods converge to the global min no matter the initialization
- **Nonconvex**: initialization matters a lot. Common heuristics: random initialization, multiple independent runs
- **Nonconvex**: clever initialization is possible with certain assumptions on the data:

https://sunju.org/research/nonconvex/

and sometimes random initialization works!

When to stop?

1st-order necessary condition: Assume f is 1st-order differentiable at $\boldsymbol{x}_0$. If $\boldsymbol{x}_0$ is a local minimizer, then $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$.
2nd-order necessary condition: Assume $f(\boldsymbol{x})$ is 2-order differentiable at $\boldsymbol{x}_0$. If $\boldsymbol{x}_0$ is a local min, $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}_0) \succeq \mathbf{0}$.

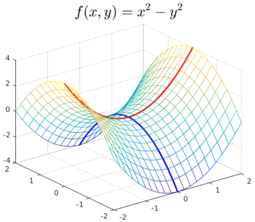
Fix some positive tolerance values ε_g, ε_H, ε_f, ε_v. Possibilities:

- $\|\nabla f(\boldsymbol{x}_k)\|_2 \le \varepsilon_g$, i.e., check 1st order cond
- $\|\nabla f(\boldsymbol{x}_k)\|_2 \le \varepsilon_g$ and $\lambda_{\min}\left(\nabla^2 f(\boldsymbol{x}_k)\right) \ge -\varepsilon_H$, i.e., check 2nd order cond
- $|f(\boldsymbol{x}_k) - f(\boldsymbol{x}_{k-1})| \le \varepsilon_f$
- $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\|_2 \le \varepsilon_v$

Nonconvex optimization is hard

Nonconvex: Even computing (verifying!) a local minimizer is NP-hard! (see, e.g., [Murty and Kabadi, 1987])

2nd order sufficient: $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}_0) \succ \mathbf{0}$
2nd order necessary: $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}_0) \succeq \mathbf{0}$

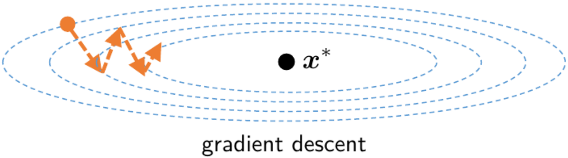


Cases in between: local shapes around SOSP determined by **spectral properties of higher-order derivative tensors**, calculating which is hard [Hillar and Lim, 2013]!

Outline

Outline

Why momentum?



Credit: Princeton ELE522

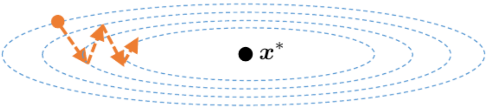
- GD is cheap ($O(n)$ per step) but overall convergence sensitive to conditioning
- Newton's convergence is not sensitive to conditioning but expensive ($O(n^3)$ per step)

A cheap way to achieve faster convergence? Answer: using historic information

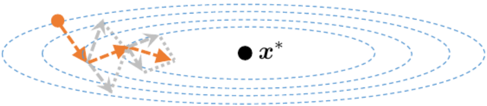
Heavy ball method

In physics, a heavy object has a large inertia/momentum — resistance to change of velocity.

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k) + \underbrace{\beta_k (\boldsymbol{x}_k - \boldsymbol{x}_{k-1})}_{\textbf{momentum}} \quad \text{due to Polyak}$$



gradient descent



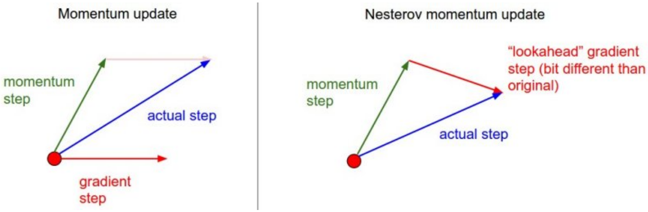
heavy-ball method

Credit: Princeton ELE522

History helps to smooth out the zig-zag path!

Another version, due to Y. Nesterov

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \beta_k \left(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\right) - \alpha_k \nabla f \left(\boldsymbol{x}_k + \beta_k \left(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\right)\right)$$



Credit: Stanford CS231N

$$\text{HB} \begin{cases} x_{\text{ahead}} = x + \beta(x - x_{\text{old}}), \\ x_{\text{new}} = x_{\text{ahead}} - \alpha \nabla f(x). \end{cases} \quad \text{Nesterov} \begin{cases} x_{\text{ahead}} = x + \beta(x - x_{\text{old}}), \\ x_{\text{new}} = x_{\text{ahead}} - \alpha \nabla f(x_{\text{ahead}}). \end{cases}$$

For more info, see Chap 10 of [Beck, 2017] and Chap 2 of [Nesterov, 2018].

Outline

Quasi-Newton methods

quasi-: seemingly; apparently but not really.

Newton's method: cost $O(n^2)$ storage and $O(n^3)$ computation per step

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t \left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$$

Idea: approximate $\nabla^2 f(\boldsymbol{x}_k)$ or $\left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1}$ to allow efficient storage and computation — **Quasi-Newton Methods**

Choose $\boldsymbol{H}_k$ to approximate $\nabla^2 f(\boldsymbol{x}_k)$ so that

- avoid calculation of second derivatives
- simplify matrix inversion, i.e., computing the search direction

Quasi-Newton methods

given: starting point $x_0 \in \operatorname{dom} f$, $H_0 \succ 0$

for $k = 0, 1, \ldots$

1. compute quasi-Newton direction $\Delta x_k = -H_k^{-1} \nabla f(x_k)$
2. determine step size t_k (*e.g.*, by backtracking line search)
3. compute $x_{k+1} = x_k + t_k \Delta x_k$
4. compute H_{k+1}

- Different variants differ on how to compute $\boldsymbol{H}_{k+1}$
- Normally $\boldsymbol{H}_k^{-1}$ or its factorized version stored to simplify calculation of $\Delta \boldsymbol{x}_k$

Credit: UCLA ECE236C

BFGS method

Broyden–Fletcher–Goldfarb–Shanno (BFGS) method

BFGS update

$$H_{k+1} = H_k + \frac{yy^T}{y^T s} - \frac{H_k s s^T H_k}{s^T H_k s}$$

where

$$s = x_{k+1} - x_k, \qquad y = \nabla f(x_{k+1}) - \nabla f(x_k)$$

Inverse update

$$H_{k+1}^{-1} = \left(I - \frac{sy^T}{y^T s}\right) H_k^{-1} \left(I - \frac{ys^T}{y^T s}\right) + \frac{ss^T}{y^T s}$$

Cost of update: $O(n^2)$ (vs. $O(n^3)$ in Newton's method), storage: $O(n^2)$ To derive the update equations, three conditions are imposed:

- secant condition: $\boldsymbol{H}_{k+1}\boldsymbol{s} = \boldsymbol{y}$ (think of 1st Taylor expansion to ∇f)
- Curvature condition: $\boldsymbol{s}_k^\mathsf{T}\boldsymbol{y}_k > 0$ to ensure that $\boldsymbol{H}_{k+1} \succ \boldsymbol{0}$ if $\boldsymbol{H}_k \succ \boldsymbol{0}$
- $\boldsymbol{H}_{k+1}$ and $\boldsymbol{H}_k$ are close in an appropriate sense

See Chap 6 of [Nocedal and Wright, 2006] Credit: UCLA ECE236C

Limited-memory BFGS (L-BFGS)

Limited-memory BFGS (L-BFGS): do not store H_k^{-1} explicitly

- instead we store up to m (*e.g.*, $m = 30$) values of

$$s_j = x_{j+1} - x_j, \qquad y_j = \nabla f(x_{j+1}) - \nabla f(x_j)$$

- we evaluate $\Delta x_k = H_k^{-1} \nabla f(x_k)$ recursively, using

$$H_{j+1}^{-1} = \left(I - \frac{s_j y_j^T}{y_j^T s_j}\right) H_j^{-1} \left(I - \frac{y_j s_j^T}{y_j^T s_j}\right) + \frac{s_j s_j^T}{y_j^T s_j}$$

  for $j = k-1, \ldots, k-m$, assuming, for example, $H_{k-m} = I$

- an alternative is to restart after m iterations

Cost of update: $O(mn)$ (vs. $O(n^2)$ in BFGS), storage: $O(mn)$ (vs. $O(n^2)$ in BFGS) — linear in dimension n! recall the cost of GD?

See Chap 7 of [Nocedal and Wright, 2006] Credit: UCLA ECE236C

Outline

Block coordinate descent

Consider a function $f(\boldsymbol{x}_1, \dots, \boldsymbol{x}_p)$ with $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, ..., $\boldsymbol{x}_p \in \mathbb{R}^{n_p}$

A generic block coordinate descent algorithm

Input: initialization $(\boldsymbol{x}_{1,0}, \dots, \boldsymbol{x}_{p,0})$ (the 2nd subscript indexes iteration number)

1: **for** $k = 1, 2, \dots$ **do**

2: Pick a block index $i \in \{1, \dots, p\}$

3: Minimize wrt the chosen block:

$$\boldsymbol{x}_{i,k} = \arg\min_{\boldsymbol{\xi} \in \mathbb{R}^{n_i}} f(\boldsymbol{x}_{1,k-1}, \dots, \boldsymbol{x}_{i-1,k-1}, \boldsymbol{\xi}, \boldsymbol{x}_{i+1,k-1}, \dots, \boldsymbol{x}_{p,k-1})$$

4: Leave other blocks unchanged: $\boldsymbol{x}_{j,k} = \boldsymbol{x}_{j,k-1} \ \forall \, j \neq i$

5: **end for**

- Also called **alternating direction/minimization methods**
- When $n_1 = n_2 = \cdots = n_p = 1$, called **coordinate descent**
- Minimization in Line 3 can be inexact: e.g.,
  $\boldsymbol{x}_{i,k} = \boldsymbol{x}_{i,k-1} - t_k \frac{\partial f}{\partial \boldsymbol{\xi}}(\boldsymbol{x}_{1,k-1}, \dots, \boldsymbol{x}_{i-1,k-1}, \boldsymbol{x}_{i,k-1}, \boldsymbol{x}_{i+1,k-1}, \dots, \boldsymbol{x}_{p,k-1})$
- In Line 2, many different ways of picking an index, e.g., cyclic, randomized, weighted sampling, etc

Block coordinate descent: examples

Least-squares $\min_{\boldsymbol{x}} \ f(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$

- $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 = \|\boldsymbol{y} - \boldsymbol{A}_{-i}\boldsymbol{x}_{-i} - \boldsymbol{a}_i x_i\|^2$
- coordinate descent: $\min_{\xi \in \mathbb{R}} \ \|\boldsymbol{y} - \boldsymbol{A}_{-i}\boldsymbol{x}_{-i} - \boldsymbol{a}_i \xi\|^2$

  $$\implies x_{i,+} = \frac{\langle \boldsymbol{y} - \boldsymbol{A}_{-i}\boldsymbol{x}_{-i}, \boldsymbol{a}_i \rangle}{\|\boldsymbol{a}_i\|_2^2}$$

  ($\boldsymbol{A}_{-i}$ is $\boldsymbol{A}$ with the i-th column removed; $\boldsymbol{x}_{-i}$ is $\boldsymbol{x}$ with the i-th coordinate removed)

Matrix factorization $\min_{\boldsymbol{A},\boldsymbol{B}} \ \|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{B}\|_F^2$

- Two groups of variables, consider block coordinate descent
- Updates:

$$\boldsymbol{A}_+ = \boldsymbol{Y}\boldsymbol{B}^\dagger,$$
$$\boldsymbol{B}_+ = \boldsymbol{A}^\dagger\boldsymbol{Y}.$$

$(\cdot)^\dagger$ denotes the matrix pseudoinverse.)

Why block coordinate descent?

- may work with constrained problems and non-differentiable problems (e.g., $\min_{\boldsymbol{A},\boldsymbol{B}} \ \|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{B}\|_F^2, \ \text{s.t.}\ \boldsymbol{A}$ orthogonal, Lasso: $\min_{\boldsymbol{x}} \ \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda \|\boldsymbol{x}\|_1$)
- may be faster than gradient descent or Newton (next)
- may be simple and cheap!

Some references:

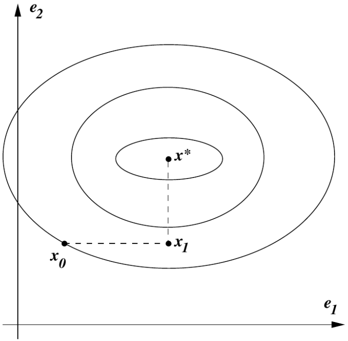
- [Wright, 2015]
- Lecture notes by Prof. Ruoyu Sun

Outline

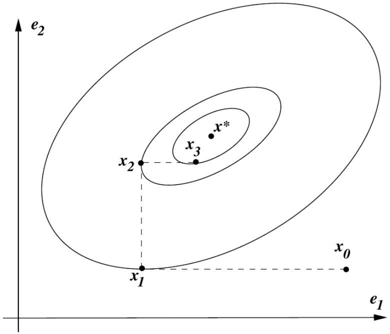
Conjugate direction methods

Solve linear equation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \iff \min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^\intercal \boldsymbol{x}$ with $\boldsymbol{A} \succ \boldsymbol{0}$

apply coordinate descent...



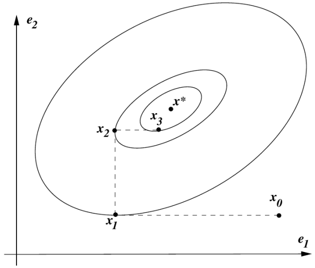
diagonal $\boldsymbol{A}$: solve the problem in n steps



non-diagonal $\boldsymbol{A}$: does not solve the problem in n steps

Conjugate direction methods

Solve linear equation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \Longleftrightarrow \min_{\boldsymbol{x}} \ \frac{1}{2}\boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^\intercal \boldsymbol{x}$ with $\boldsymbol{A} \succ \boldsymbol{0}$



non-diagonal $\boldsymbol{A}$: does not solve the problem in n steps

Idea: define n "conjugate directions" $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\}$ so that $\boldsymbol{p}_i^\intercal \boldsymbol{A} \boldsymbol{p}_j = 0$ for all $i \neq j$—conjugate as generalization of orthogonal

- Write $\boldsymbol{P} = [\boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$. Can verify that $\boldsymbol{P}^\intercal \boldsymbol{A} \boldsymbol{P}$ is diagonal and positive
- Write $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{s}$. Then $\frac{1}{2}\boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^\intercal \boldsymbol{x} = \frac{1}{2}\boldsymbol{s}^\intercal \left(\boldsymbol{P}^\intercal \boldsymbol{A} \boldsymbol{P}\right) \boldsymbol{s} - \left(\boldsymbol{P}^\intercal \boldsymbol{b}\right)^\intercal \boldsymbol{s}$ — quadratic with diagonal $\boldsymbol{P}^\intercal \boldsymbol{A} \boldsymbol{P}$
- Perform updates in the $\boldsymbol{s}$ space, but write the equivalent form in $\boldsymbol{x}$ space
- The i-the coordinate direction in the $\boldsymbol{s}$ space is $\boldsymbol{p}_i$ in the $\boldsymbol{x}$ space

In short, change of variable trick!

Conjugate gradient methods

Solve linear equation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \Longleftrightarrow \min_{\boldsymbol{x}} \ \frac{1}{2}\boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^\intercal \boldsymbol{x}$ with $\boldsymbol{A} \succ \boldsymbol{0}$
Idea: define n "conjugate directions" $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\}$ so that $\boldsymbol{p}_i^\intercal \boldsymbol{A}\boldsymbol{p}_j = 0$ for all $i \neq j$—conjugate as generalization of orthogonal

Generally, many choices for $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\}$.
Conjugate gradient methods: choice based on ideas from steepest descent

Algorithm 5.2 (CG).
Given x_0;
Set $r_0 \leftarrow Ax_0 - b$, $p_0 \leftarrow -r_0$, $k \leftarrow 0$;
while $r_k \neq 0$

$$\alpha_k \leftarrow \frac{r_k^T r_k}{p_k^T A p_k}; \tag{5.24a}$$

$$x_{k+1} \leftarrow x_k + \alpha_k p_k; \tag{5.24b}$$

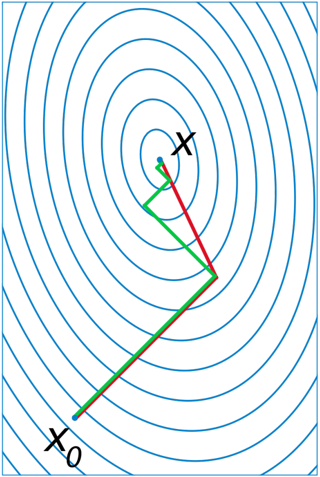
$$r_{k+1} \leftarrow r_k + \alpha_k A p_k; \tag{5.24c}$$

$$\beta_{k+1} \leftarrow \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}; \tag{5.24d}$$

$$p_{k+1} \leftarrow -r_{k+1} + \beta_{k+1} p_k; \tag{5.24e}$$

$$k \leftarrow k + 1; \tag{5.24f}$$

end (while)



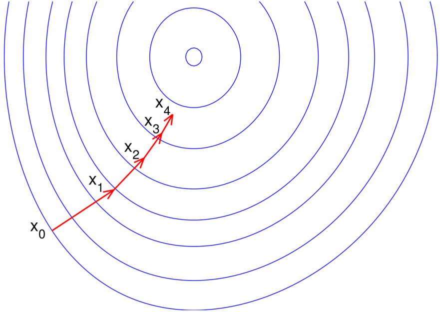
CG vs. GD (Green: GD, Red: CG)

- Can be extended to general non-quadratic functions
- Often used to solve subproblems of other iterative methods, e.g., truncated Newton method, the trust-region subproblem (later)

See Chap 5 of [Nocedal and Wright, 2006]

Outline

Iterative methods



Credit: aria42.com

Illustration of iterative methods on the contour/levelset plot (i.e., the function assumes the same value on each curve)

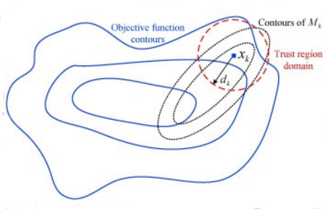
Two questions: what direction to move, and how far to move

Two possibilities:

- **Line-search methods**: direction first, size second
- **Trust-region methods** (TRM): size first, direction second

Ideas behind TRM

Recall Taylor expansion $f(\boldsymbol{x}+\boldsymbol{d}) \approx f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}\rangle + \frac{1}{2}\langle \boldsymbol{d}, \nabla^2 f(\boldsymbol{x}_k)\boldsymbol{d}\rangle$



Credit: [Arezki et al., 2018]

Start with $\boldsymbol{x}_0$. Repeat the following:

- At $\boldsymbol{x}_k$, approximate f by the quadratic function (called model function dotted black in the left plot)

$$m_k(\boldsymbol{d}) = f(\boldsymbol{x}_k) + \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d}\rangle + \frac{1}{2}\langle \boldsymbol{d}, \boldsymbol{B}_k \boldsymbol{d}\rangle$$

  i.e., $m_k(\boldsymbol{d}) \approx f(\boldsymbol{x}_k + \boldsymbol{d})$, and $\boldsymbol{B}_k$ to approximate $\nabla^2 f(\boldsymbol{x}_k)$

- Minimize $m_k(\boldsymbol{d})$ within a **trust region** $\{\boldsymbol{d} : \|\boldsymbol{d}\| \le \Delta\}$, i.e., a norm ball (in red), to obtain $\boldsymbol{d}_k$
- If the approximation is inaccurate, decrease the region size; if the approximation is sufficiently accurate, increase the region size.
- If the approximation is reasonably accurate, update the iterate $\boldsymbol{x}_{k+1} = x_k + \boldsymbol{d}_k$.

Framework of trust-region methods

To measure approximation quality: $\rho_k \doteq \frac{f(\boldsymbol{x}_k) - f(\boldsymbol{x}_k + \boldsymbol{d}_k)}{m_k(\boldsymbol{0}) - m_k(\boldsymbol{d}_k)} = \frac{\text{actual decrease}}{\text{model decrease}}$

A generic trust-region algorithm

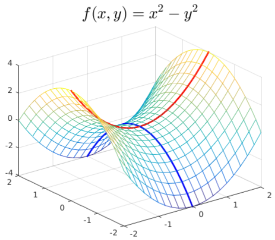
Input: $\boldsymbol{x}_0$, radius cap $\widehat{\Delta} > 0$, initial radius Δ_0, acceptance ratio $\eta \in [0, 1/4)$

for $k = 0, 1, \ldots$ **do**
    $\boldsymbol{d}_k = \arg\min_{\boldsymbol{d}} m_k(\boldsymbol{d})$, s. t. $\|\boldsymbol{d}\| \le \Delta_k$ **(TR Subproblem)**
    if $\rho_k < 1/4$ **then**
        $\Delta_{k+1} = \Delta_k/4$
    else
        if $\rho_k > 3/4$ and $\|\boldsymbol{d}_k\| = \Delta_k$ **then**
            $\Delta_{k+1} = \min\left(2\Delta_k, \widehat{\Delta}\right)$
        else
            $\Delta_{k+1} = \Delta_k$
        end if
    end if
    if $\rho_k > \eta$ **then**
        $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k$
    else
        $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k$
    end if
end for

Why TRM?

Recall the model function $m_k(\boldsymbol{d}) \doteq f(\boldsymbol{x}_k) + \langle \nabla f(\boldsymbol{x}_k), \boldsymbol{d} \rangle + \frac{1}{2} \langle \boldsymbol{d}, \boldsymbol{B}_k \boldsymbol{d} \rangle$

- Take $\boldsymbol{B}_k = \nabla^2 f(\boldsymbol{x}_k)$
- Gradient descent: stop at $\nabla f(\boldsymbol{x}_k) = \mathbf{0}$
- Newton's method: $\left[\nabla^2 f(\boldsymbol{x}_k)\right]^{-1} \nabla f(\boldsymbol{x}_k)$ may just stop at $\nabla f(\boldsymbol{x}_k) = \mathbf{0}$ or be ill-defined
- Trust-region method: $\min_{\boldsymbol{d}} \; m_k(\boldsymbol{d}) \quad \text{s. t.} \; \|\boldsymbol{d}\| \le \Delta_k$



When $\nabla f(\boldsymbol{x}_k) = \mathbf{0}$,

$$m_k(\boldsymbol{d}) - f(\boldsymbol{x}_k) = \frac{1}{2} \left\langle \boldsymbol{d}, \nabla^2 f(\boldsymbol{x}_k) \boldsymbol{d} \right\rangle .$$

If $\nabla^2 f(\boldsymbol{x}_k)$ has negative eigenvalues, i.e., there are negative directional curvatures, $\frac{1}{2} \left\langle \boldsymbol{d}, \nabla^2 f(\boldsymbol{x}_k) \boldsymbol{d} \right\rangle < 0$ for certain choices of $\boldsymbol{d}$ (e.g., eigenvectors corresponding to the negative eigenvalues)

TRM can help to move away from "nice" saddle points!

To learn more about TRM

- A comprehensive reference [Conn et al., 2000]
- A closely-related alternative: cubic regularized second-order (CRSOM) method [Nesterov and Polyak, 2006, Agarwal et al., 2018]
- Example implementation of both TRM and CRSOM: Manopt (in Matlab) `https://www.manopt.org/` (choosing the Euclidean manifold)

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