1D phase retrieval and spectral factorization

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Joint work with Ju Sun

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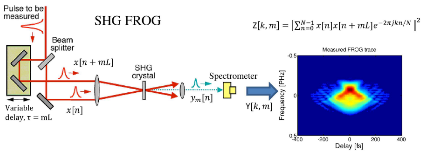
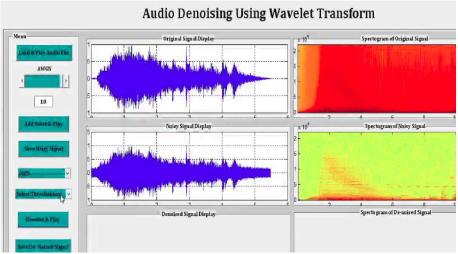
1d phase retrieval

Given the oversampled Fourier transform magnitudes $|\mathcal{F}(x)|^2$, recover x.



Frequency-resolved optical gating (FROG)

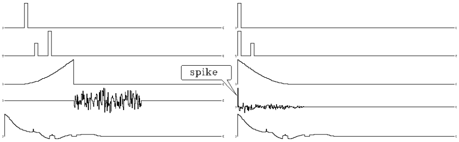


Wavelet-based speech processing

Spectral Factorization

- Recall: multiplication $\underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftharpoons}}$ convolution
- $|\mathcal{F}(x)|^2 \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftharpoons}} x \star x$

$$\text{PR} : |\mathcal{F}(x)|^2 \mapsto x \iff \text{spectral factorization} : x \star x \mapsto x$$



Non-uniqueness

- Usually no unique solution (very different than 2D case!)
- Eg:

$$(6, 5, -1) \star (6, 5, -1) = (6, -35, 62, -35, 6)$$
$$(3, -7, 2) \star (3, -7, 2) = (6, -35, 62, -35, 6)$$

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$$6 + 5z - z^2 = (z - 2)(z - 3) \rightarrow \text{Roots} : r_1 = 2, r_2 = 3$$
$$3 - 7z + 2z^2 = 2(z - 1/2)(z - 3) \rightarrow \text{Roots} : r_1 = 1/2, r_2 = 3$$

Non-uniqueness

In general, all solutions obtained by flipping roots in the unit circle, $r \mapsto \bar{r}^{-1}$.

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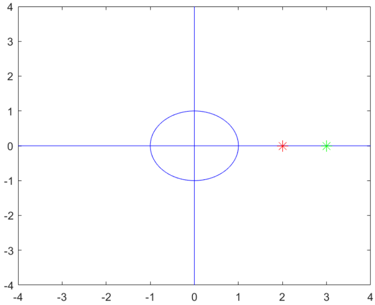

Non-uniqueness

In general, all solutions obtained by inverting roots in the unit circle, $r \mapsto \bar{r}^{-1}$.

$$6 + 5z - z^2 = (z-2)(z-3) \to \text{Roots}: r_1 = 2, r_2 = 3$$

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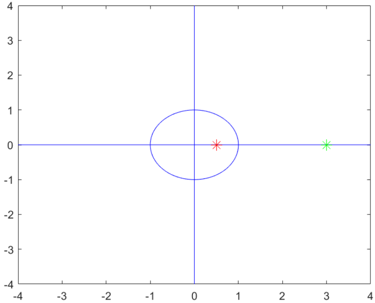

Non-uniqueness

- In general, for a signal $x \in \mathbb{R}^n$, up to 2^n possible solutions. (One for each root inversion.)
- Special case: $Z\{x\}$ has all its roots on the unit circle. $\rightarrow$ Unique solution!

Least-squares minimization

Given $r = x \star x$, solve:

$$\min_x \; f(x) = \frac{1}{2}\|r - x \star x\|^2. \tag{1.1}$$

Least-squares minimization

Given $r = x \star x$, solve:

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- Nonconvex function ($\Rightarrow$ in general NP-hard to find global minimum)
- Can we achieve success?

Why we care about least-squares

- When any feasible global minimizer is acceptable, or x is root-unitary.

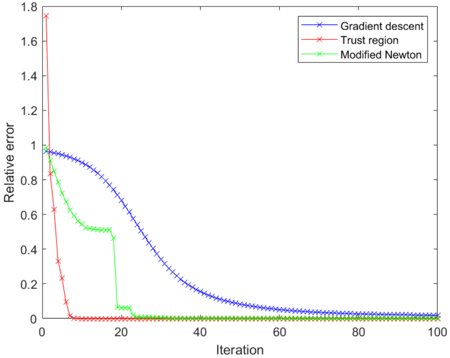
- When we can regularize to a particular solution (add priors).
  - Sparsity (l_1-regularization)
  - Symmetry
  - Known support

History: Wilson's method for spectral factorization

- (Wilson, 1969): Start from $x = [1; 0, \dots, 0]$ and apply Newton-Raphson to find the roots of $g(x) = r - x \star x$.
- Surprise: guaranteed recovery!
- **Observation:** Newton-Raphson is equivalent to Gauss-Newton on a least-squares objective.

Is Gauss-Newton special?

- Many numerical optimization algorithms:
  - Gradient descent
  - Trust region
  - Modified Newton
  - Many others...
- $\Rightarrow$ All achieve success!



Case 1: the ground-truth x^* is Gaussian random.

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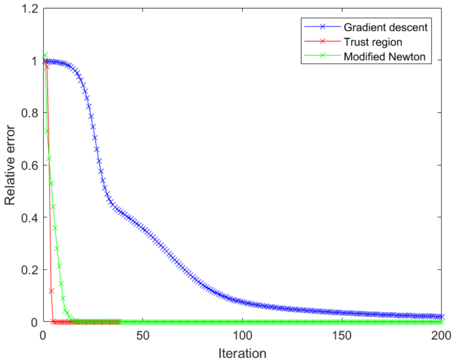
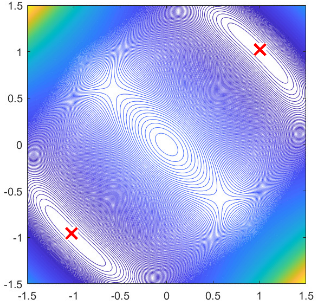
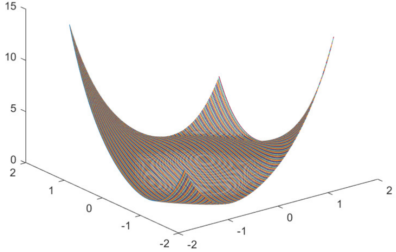


Case 2: the ground-truth x^* has all its roots on the unit circle.
(Problematic for other methods.)

Global landscape analysis

Landscape determines algorithm behavior





Recurrent theme in nonconvex optimization (e.g. Sun et. al., 2018)

Open problems

- Function landscape analysis
- 2D least-squares (new initialization)