Low-Rank Representation with Positive SemiDefinite Constraint (LRR-PSD)

-- A Robust Approach for Subspace Segmentation

Xiaotong Yuan

Joint work with **Ju Sun, Yuzhao Ni, Guangcan Liu, Shuicheng Yan, and Loong-Fah Cheong**
National University of Singapore

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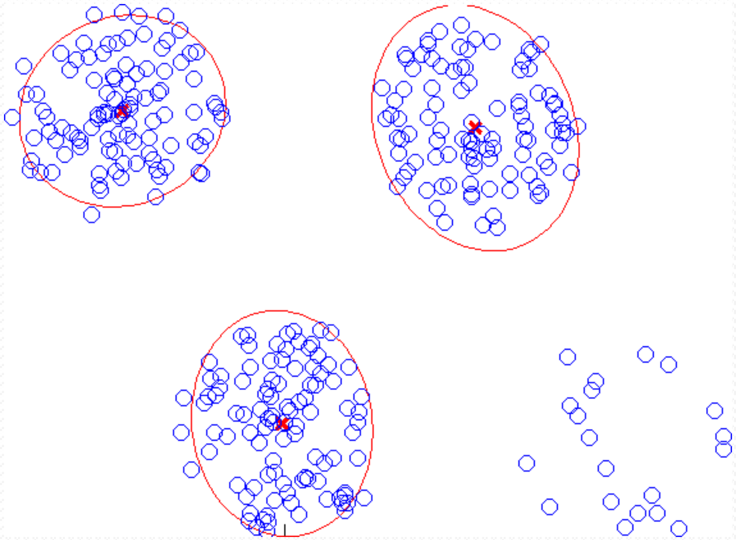
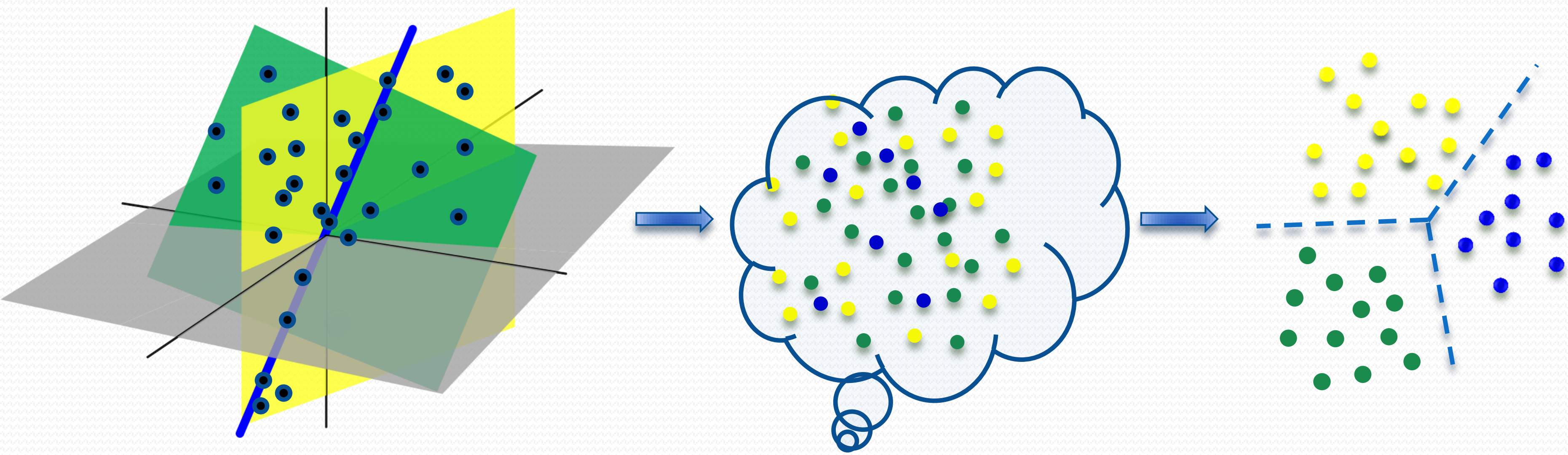
Less Structured Clustering …



Some popular clustering algorithms

- **Kmeans**
- **Mean-Shift (mode-seeking)**
- **Mixture models (e.g., GMM)**
- **Hierarchical methods**
- **…**
- **Spectral clustering**

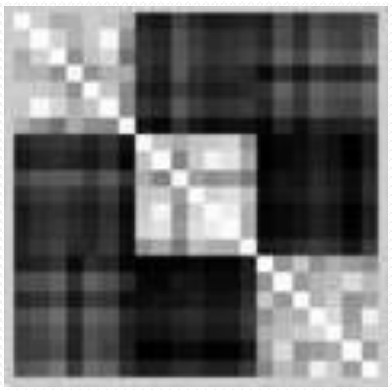
Subspace Clustering



Data generation: Sampling

Segmentation/Clustering

Spectral Clustering



$$\mathbf{D} = Diag(\mathbf{W}\mathbf{1})$$

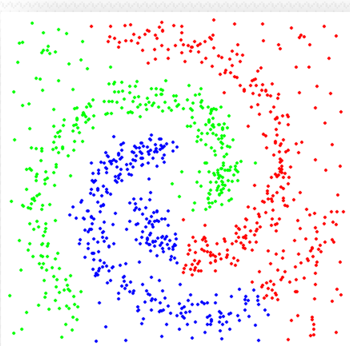
$$\mathbf{L} = I - \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{-1/2}$$

$$\mathbf{L}\mathbf{V} = \mathbf{V}\Lambda$$

Affinity matrix W

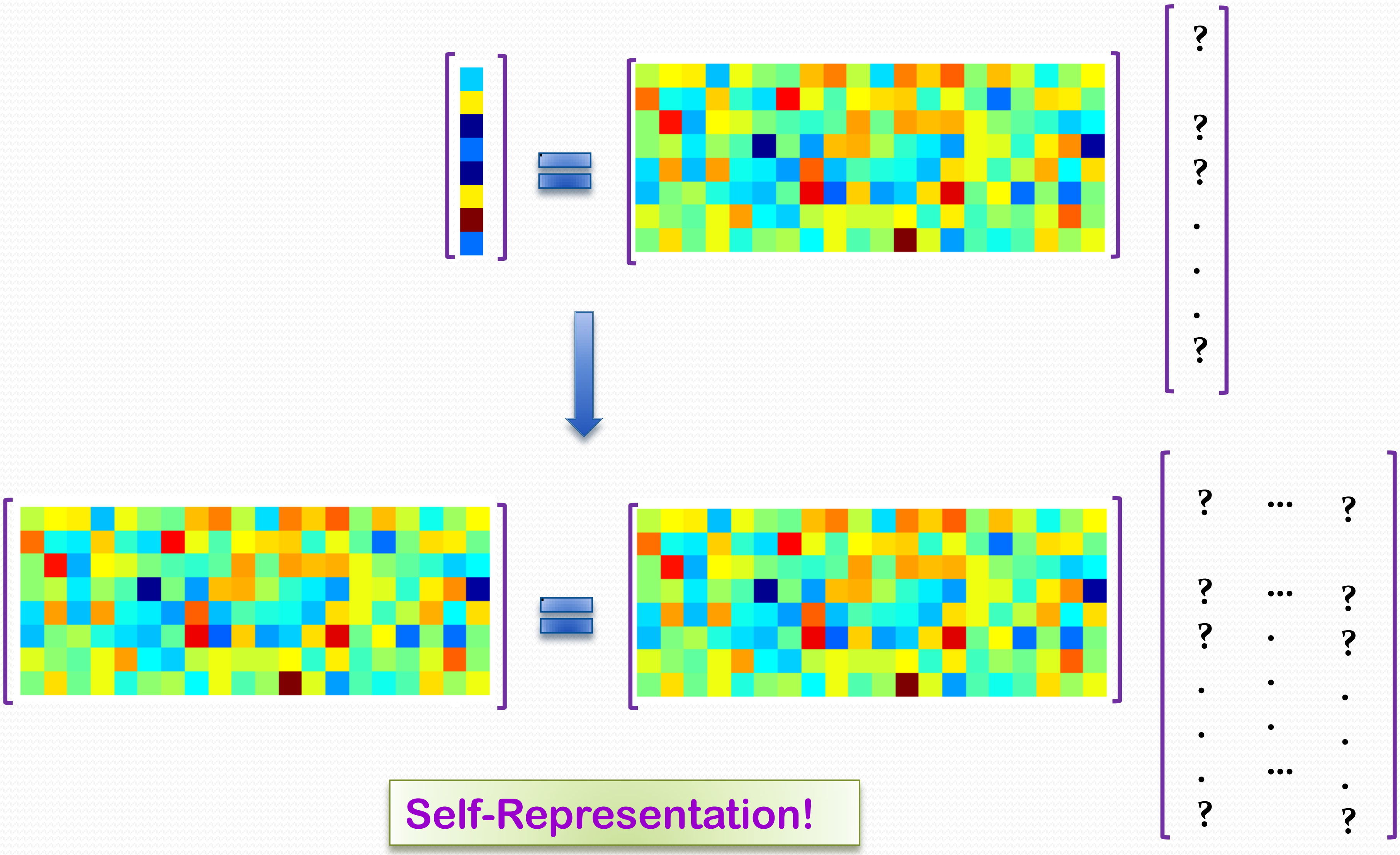
Laplacian matrix L

Eigen-analysis

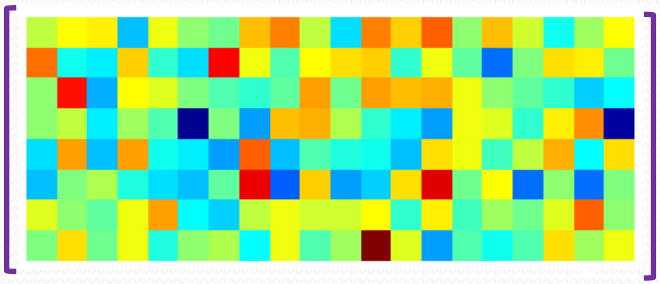


Ideal W shall be
1. Block-diagonal
2. Positive semi-definite, i.e. $\mathbf{W} \succeq \mathbf{0}$

Learning the Affinity Matrix



Low Rank Representation



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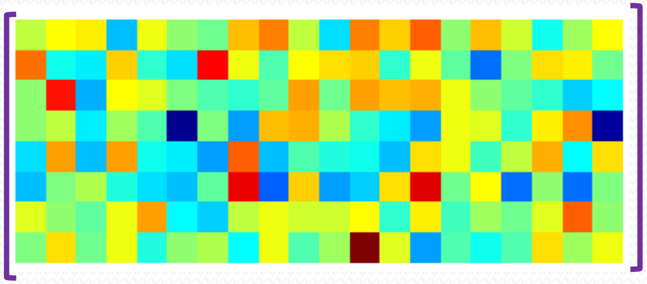


$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ \vdots & \vdots & \vdots \\ ? & ? & ? \end{bmatrix}$$

$$\mathbf{X} = \mathbf{XZ}$$

Trivial Solution: I

Low-Rank Objective

$$\min. \quad \text{rank} \ \mathbf{Z}$$
$$\text{subj.} \quad \mathbf{X} = \mathbf{XZ}$$

→

Convex Relaxation

$$\min. \quad \|\mathbf{Z}\|_*$$
$$\text{subj.} \quad \mathbf{X} = \mathbf{XZ}$$

G. Liu *et al.*, ICML 2010

Blessings of the Learned Affinity

LRR		LRR-PSD
$\min.\ \|\mathbf{Z}\|_*,$ $\text{subj.}\ \mathbf{X} = \mathbf{XZ}$	$\Leftrightarrow$	$\min.\ \|\mathbf{Z}\|_*,$ $\text{subj.}\ \mathbf{X} = \mathbf{XZ}, \mathbf{Z} \succeq \mathbf{0}.$

Blessing (1) The minimizers to both are unique and identical!

Blessing (2) Z will be block-diagonal for sorted data!

Blessing (3) The optimal Z will always be positive semi-definite!

In the presence of noise/outliers, a robust formulation

$$\min.\ \|\mathbf{Z}\|_* + \|\mathbf{E}\|_{2,1},\ \text{subj.}\ \mathbf{X} = \mathbf{XZ} + \mathbf{E}, \mathbf{Z} \succeq \mathbf{0}.$$

Augmented Lagrange Multiplier (ALM) Method

ALM – An Interpolation of Lagrange Form and Penalty Form

$$\begin{aligned} \min. \quad & f(X), \\ \text{subj.} \quad & h(X) = 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} & L(X, Y, \mu) \\ & = f(X) + \langle Y, h(x) \rangle + \frac{\mu}{2} \| h(X) \|_F^2 \end{aligned}$$

Lagrange Form

Penalty Form

For increasing μ

1) Minimize L wrt. X

2) Update the multiplier Y (dual ascent)

In passing μ to infinity, the ALM form solves the original program.

Augmented Lagrange Multiplier (ALM) Method

$$\min.\ \|\mathbf{Z}\|_* + \|\mathbf{E}\|_{2,1},\ \text{subj.}\quad \mathbf{X} = \mathbf{X}\mathbf{Z} + \mathbf{E}, \mathbf{Z} \succeq \mathbf{0}.$$

$$\Downarrow$$

$$\min.\ \|\mathbf{J}\|_* + \|\mathbf{E}\|_{2,1},\ \text{subj.}\quad \mathbf{X} = \mathbf{X}\mathbf{Z} + \mathbf{E}, \mathbf{J} = \mathbf{Z}, \mathbf{Z} \succeq \mathbf{0}.$$

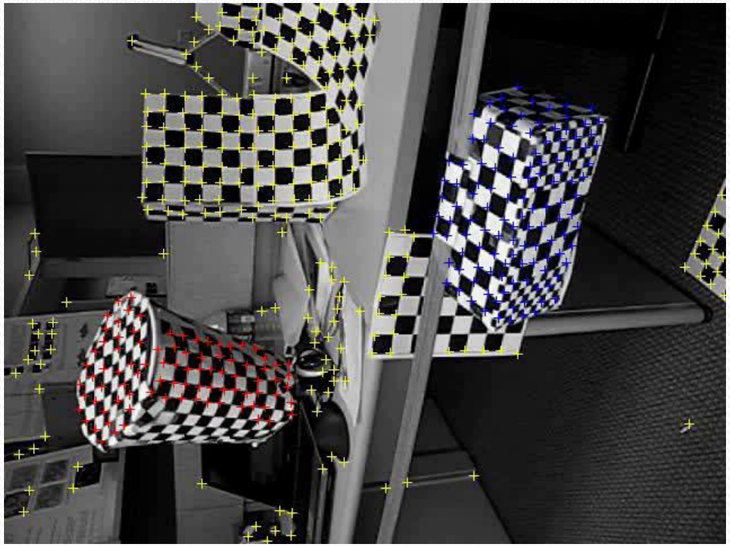
$$\Downarrow$$

$$L(\mathbf{Z}, \mathbf{E}, \mathbf{J}, \mathbf{Y}_1, \mathbf{Y}_2, \mu)$$
$$= \|\mathbf{J}\|_* + \lambda\|\mathbf{E}\|_{2,1} + \langle \mathbf{Y}_1, \mathbf{X} - \mathbf{X}\mathbf{Z} - \mathbf{E}\rangle + \langle \mathbf{Y}_2, \mathbf{Z} - \mathbf{J}\rangle$$
$$+ \frac{\mu}{2}\|\mathbf{X} - \mathbf{X}\mathbf{Z} - \mathbf{E}\|_F^2 + \frac{\mu}{2}\|\mathbf{Z} - \mathbf{J}\|_F^2$$

Optimizing wrt. Z, E, J has simple closed-form solutions.

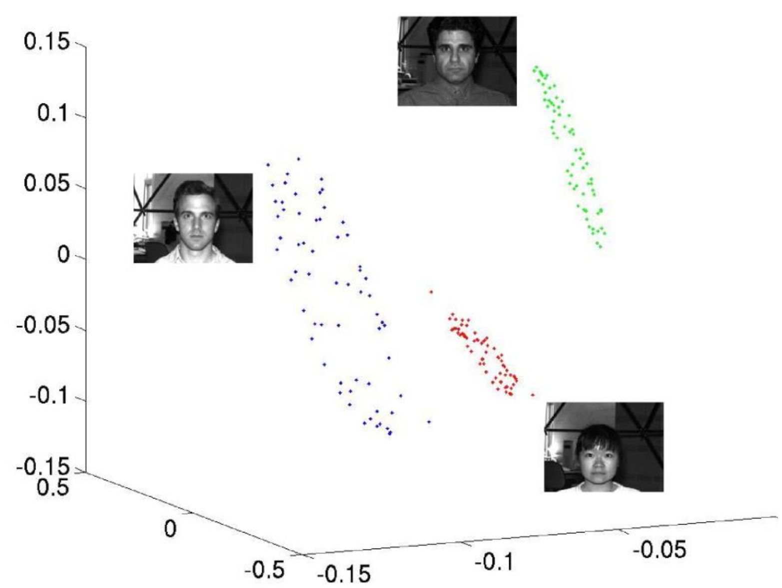
Application I – Motion Segmentation

Grouping of motion trajectories according to motion patterns.



Application II – Face Clustering

Extensions to segment data lying on low-rank manifolds



Face Manifolds

Example faces from Extened Yale B Face Dataset (EYB)

Table: Segmentation accuracy (%) on EYB. We record the average performance from multiple runs instead of the best.

	Gauss SC	Linear SC	SSC	LRR	LRR-PSD
Acc.	24.84	30.16	37.66	**59.53**	**60**

Summary

1. Clustering structured data invite more elegant solutions.

2. Performance guaranteed by learning affinity matrices for subspace clustering

3. An efficient optimization strategy based on ALM

4. Applications in motion segmentation and face image clustering: possible to extend to low-dimensional manifolds that behave similarly locally as subspaces

Some Results and Recent Developments

$$\begin{aligned} \min.\ & \|\mathbf{Z}\|_*, \\ \text{subj.}\ & \mathbf{X} = \mathbf{XZ} \end{aligned} \quad \Rightarrow \mathbf{Z}^* = \mathbf{V}\mathbf{V}^\top, \text{for } \mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$$

$$\begin{aligned} \min.\ & \|\mathbf{Z}\|_*, \\ \text{subj.}\ & \mathbf{X} = \mathbf{AZ} \\ & \mathbf{X} \in \text{span}(\mathbf{A}) \end{aligned} \quad \begin{aligned} & \Rightarrow \mathbf{Z}^* = \mathbf{V}_A \left(\mathbf{V}_A^\top \mathbf{V}_A\right)^{-1} \mathbf{V}_X^\top, \\ & \text{for } [\mathbf{X}, \mathbf{A}] = \mathbf{U}\mathbf{D}[\mathbf{V}_X^\top, \mathbf{V}_A^\top] \end{aligned}$$

These solutions are unique!

References

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Thank you for your attention!

Questions?