1D Phase Retrieval and Spectral Factorization

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Abstract: The 1D phase retrieval problem can be solved via several optimization methods on a least-squares formulation. This provides a new algorithmic toolbox and asks for new theoretical insight.

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1. 1D phase retrieval and spectral factorization

The classic 1D *phase retrieval* (PR) problems concerns recovering a 1D signal x from its oversampled Fourier magnitudes $|\boldsymbol{F}(x)|^2$. This is equivalent to recovering x from its autocorrelation sequence $r = x \star x$ — the two are linked by an inverse Fourier transform. In general, the problem may or may not have a unique solution (we always ignore the trivial symmetries), determined by the root distribution of the polynomial $\boldsymbol{Z}(x)$ ($\boldsymbol{Z}(\cdot)$ denotes the $\boldsymbol{Z}$ transform) on or off the complex unit circle [1]. In *spectral factorization* (SP), which arises in digital signal processing and control [2], one is given $\boldsymbol{Z}(r)$, and the task is to find a factorization $\boldsymbol{Z}(r) = \alpha F(z)F(z^{-1})$ so that $F(z)$ has all its roots inside the unit circle (called the *minimum-phase factor*). Contrasting the 1D PR problem, the solution to the SP problem is always unique, if any. Numerous methods have been derived to solve the SP problem, as summarized in [2].

2. A least-squares formulation

Both problems can be cast into a natural least-squares formulation

$$\min_x \; f(x) = \frac{1}{2}\|r - x \star x\|^2. \tag{2.1}$$

For the PR problem, let us be content with finding any feasible x that makes the objective f zero, i.e., solving $r = x \star x$.

2.1. *A remarkably simple method for SP*

Setting up the problem as root finding to the quadratic function $g(x) = r - x \star x$, Wilson [3] proved that starting from $x = [1; 0, \ldots, 0]$ and applying the classic Newton-Raphson iteration $x^{k+1} = x^k - J^{-1}(x^k)g(x^k)$ ($J(x^k)$ is the Jacobian at x^k) return the unique SP solution, if one exists (see an improved analysis in [4]).

This method is simple and natural, and the result is powerful. Particularly, the direct connection of Newton-Raphson method for root finding and the Gauss-Newton's method for solving nonlinear least-squares in numerical optimization [5] implies that: *the above iterative scheme also provably solves the least-squares formulation* (2.1) *for 1D PR*. This should comes as a surprise, as (2.1) is a nonconvex optimization problem and obtaining a global minimum for nonconvex problems is generally difficult [6].

2.2. *Why we like the least-squares, still?*

Up to this point, the central 1D PR problem seems already well solved. However, we argue that the least-squares formulation (2.1) deserves a through study, because (1) additional priors about x, e.g., sparsity, can be easily incorporated into the optimization framework, but not the nonlinear equations; (2) many other numerical methods can be applied to solve (2.1), which offers additional algorithmic flexibility, especially when additional priors are included, and (3) mathematically, the proof in [3, 4] is tied to the Newton-Raphson's iteration and not much intuition can be extracted as for why the method would work successfully. The least-squares might offer some insights to this, as we show below.

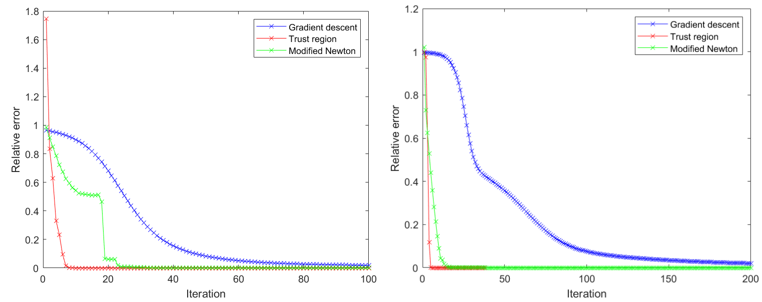

Fig. 1: Plot of the relative data error $\frac{\|x\star x - x^*\star x^*\|}{\|x^*\star x^*\|}$ vs. iterate for various optimization algorithms applied to the least-squares objective function (2.1). (Left) shows the case of a random, Gaussian ground truth signal x^*, and (Right) shows the case where $\boldsymbol{Z}(x^*)$ has all its roots on the unit circle.

2.3. *What methods work for the least-squares?*

As a preliminary study, we examine what numerical methods can solve (2.1), in the sense of giving zero objective value. We employ three additional numerical methods: gradient descent, trust region, and modified Newton. All of them are very standard numerical methods in solving smooth optimization problems and can be found in general optimization textbooks [5]. For all methods, we follow [3] and initialize x as $[1;0,\dots,0]$. We test on two cases: (1) the ground-truth x^* is Gaussian random, i.e., there are many other global solutions to (2.1) other than x^* (2) $\boldsymbol{Z}(x^*)$ of the ground-truth x^* has all its roots on the unit circle, i.e., the solution to (2.1) is unique (again, up to the trivial symmetries). In both cases, all three methods we test find a global solution to the nonconvex problem (2.1), as shown in Fig. 1. We have gathered highly consistent numerical results on many other examples than shown here.

3. Towards understanding from the local function landscape

Our experimental results imply that in terms of solving (2.1) for 1D PR, there is nothing special about Newton-Raphson (Gauss-Newton) and its analysis. We conjecture that the nice local behaviors of the various numerical methods are determined by an associated nice local function landscape. Indeed, such nice local landscape has been proved for Gaussian PR and masked Fourier PR [7, 8]. We believe similar characterization will impact theory and practice of PR.

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References

1. R. Beinert and G. Plonka, “Ambiguities in one-dimensional discrete phase retrieval from fourier magnitudes,” Journal of Fourier Analysis and Applications **21**, 1169–1198 (2015).
2. A. H. Sayed and T. Kailath, “A survey of spectral factorization methods,” Numerical Linear Algebra with Applications **8**, 467–496 (2001).
3. G. Wilson, “Factorization of the covariance generating function of a pure moving average process,” SIAM Journal on Numerical Analysis **6**, 1–7 (1969).
4. H. Orchard and A. Willson, “On the computation of a minimum-phase spectral factor,” IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications **50**, 365–375 (2003).
5. P. E. Gill, W. Murray, and M. H. Wright, “Practical optimization,” (1981).
6. J. Sun, Q. Qu, and J. Wright, “When are nonconvex problems not scary?” arXiv preprint arXiv:1510.06096 (2015).
7. E. J. Candes, X. Li, and M. Soltanolkotabi, “Phase retrieval via wirtinger flow: Theory and algorithms,” IEEE Transactions on Information Theory **61**, 1985–2007 (2015).
8. J. Sun, Q. Qu, and J. Wright, “A geometric analysis of phase retrieval,” Foundations of Computational Mathematics (2017).